Sample final project: One signature of chaos in the double pendulum

Alex Small*

Department of Physics and Astronomy,

California State Polytechnic University, Pomona, CA 91768

Abstract

The double pendulum is one of the simplest and most intuitive examples of chaos in a Hamiltonian system. There are many ways in which the chaotic behavior of this system can be quantified and illustrated. One especially interesting way is to look for a qualitative behavior (such as the bottom half of the pendulum flipping over, *i.e.* rotating through 180 degrees), and ask how much time it takes to observe this behavior for different initial conditions. An article on the very popular and sometimes reliable Wikipedia claims that the qualitative behavior of the double pendulum is indeed chaotic. Specifically, the article claims that the time required to flip over depends on initial conditions in a manner that exhibits fractal patterns. Our purpose here is to "audit" Wikipedia, and determine if this claim is true. We will also investigate whether/how any fractal patterns depend on basic parameters of the pendulum (*e.g.* mass and length ratios).

I. BACKGROUND

Chaotic motion is a common phenomenon in mechanical systems. Generally speaking, if a system has more than 1-2 degrees of freedom, nonlinear equations of motion, and relatively few symmetries, the motion of the system is difficult to predict. Small changes in initial conditions can lead to large quantitative and qualitative changes in the long-term behavior of the system[1]. Mere difficulty of prediction, however, is not the most intellectually fascinating aspect of chaotic motion; it is unsurprising that systems with many degrees of freedom and few simplifying symmetries should be difficult to model. Rather, what is most fascinating about chaotic motion is that in spite of its unpredictable nature it nonetheless exhibits patterns. Typically, these patterns are not found by looking at individual trajectories, but rather by comparing some property of different trajectories (*e.g.* bifurcation diagrams, chaotic attractors) or by comparing different points along a single trajectory (*e.g.* Poincare sections). In particular, the Poincare sections and bifurcation diagrams of chaotic systems often exhibit fractal patterns, *i.e.* patterns that are self-similar, looking the same at different levels of magnification[2]

One commonly-studied chaotic system is the double-pendulum (Fig. 1, from Wolfram Scienceworld[11]). Consisting of two pendula attached to each other and allowed to swing freely without friction, the double pendulum is of wide interest because it is one of the simplest systems to exhibit chaotic behavior while conserving energy. It is easier to model numerically than the well-known 3-body gravitational problem (which is arguably one of the starting points for the theory of chaos, in Poincare's work on the solar system), feasible to implement in a low-friction experiment[3,4], and easy to visualize. Most interestingly, the sensitivity to initial conditions can apparently be illustrated graphically, in a manner that also reveals fractal behavior. An article on Wikipedia[12] (admittedly not always the most reliable of sources) includes a figure (reproduced in Fig. 2) showing how the "flip time" depends on initial conditions. The flip time is defined as the amount of time that it takes for the bottom pendulum to flip upside down, *i.e.* reach $\theta_2 = \pm\pi$. Interestingly, there are dense sets of initial conditions that have sufficient energy to flip upside down, but do not flip during the duration of the (long) simulations described on Wikipedia. Moreover, a contour plot of flip time *vs.* θ_1 and θ_2 exhibits an apparent fractal structure.

While this figure on Wikipedia is not from a peer-reviewed source, it is analogous to an

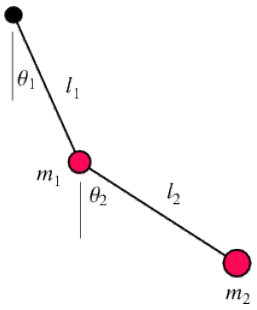

FIG. 1: (Taken from Wolfram Scienceworld) A schematic diagram of the double pendulum, consisting of two masses m_1 and m_2, connected to massless rods. The dynamical variables that we will model are θ_1 and θ_2, defined in the figure.

interesting peer-reviewed finding on chaos theory: the problem of escape velocity from the solar system[5]. Escape velocity is trivial to calculate if one is escaping from the gravitational field of an isolated mass, but as soon as multiple objects are included (*i.e.* as soon as we consider the very realistic case of launching from the surface of earth while trying to escape

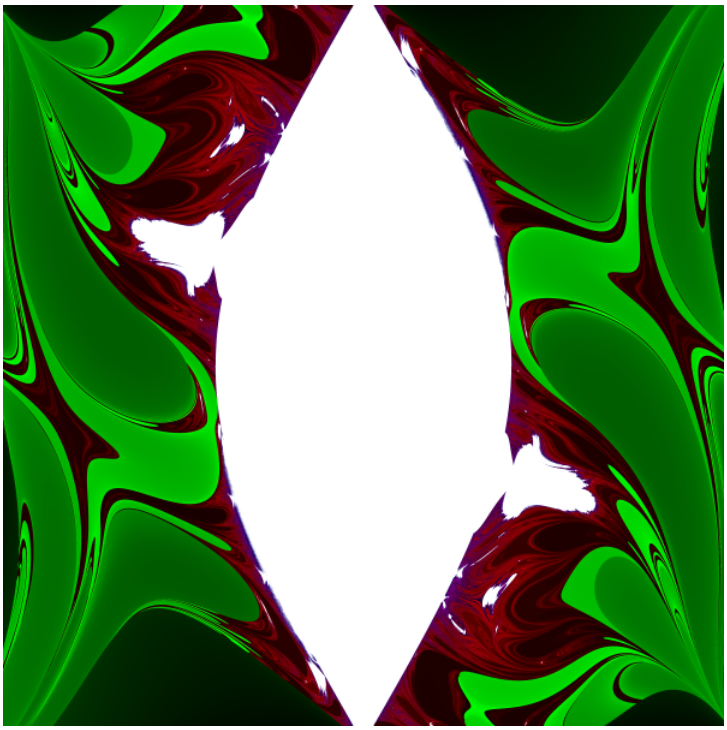

FIG. 2: (Taken from Wikipedia) Flip time *vs.* starting angles (which range from -3 radians to $+3$ radians, θ_1 being on the horizontal axis and θ_2 on the vertical) for a double pendulum composed of two rigid bars of equal mass m and length ℓ. Green denotes a flip time of less than $10\sqrt{\ell/g}$, red denotes a flip time between $10\sqrt{\ell/g}$ and $10^2\sqrt{\ell/g}$, purple denotes a flip time between $10^2\sqrt{\ell/g}$ and $10^3\sqrt{\ell/g}$, and blue denotes a flip time between $10^3\sqrt{\ell/g}$ and $10^4\sqrt{\ell/g}$. No flip occurs in the white regions.

the sun's gravity) we have a 3-body (or more) problem, and figuring out the minimum launch speed becomes rather complicated. It is admittedly one of the simpler 3-body problems, since the mass of the rocket will not have any measurable effect on the trajectories of the earth and sun, but the potential experienced by the rocket is now time-dependent (because there is no inertial frame in which all of the large objects are stationary) and considerably more complicated than the simple potentials typically studied in textbooks. If one simulates the motion of a rocket launched in different directions and at different speeds, and computes the ultimate distance d from the sun as a function of launch parameters, a contour plot of d *vs.* launch parameters exhibits fractal structure[5]. Consequently, it is very plausible that the results in the Wikipedia figure are valid. Our goal here is to determine if those results can be reproduced in a similar context, namely a double pendulum consisting of two point masses joined by massless rods.

II. METHODS

The equations of motion are derived in the Appendix. In order to simplify the equations, we set the parameters m_1 and l_1 equal to 1, to fix our units of mass and length. We chose units of time such that $g = 4\pi^2$, so that small oscillations of the top pendulum would have period $T = 1$. We solved the differential equations with a standard workhorse algorithm for solving ordinary differential equations: fourth-order Runge-Kutta with an adaptive step-size[6]. We also attempted to use Euler-Cromer[7] with an adaptive step size, but found that consistent energy conservation was difficult to achieve for any reasonable step size. (Data not shown.) This is a consequence of the fact that explicit symplectic integrators generally only work for systems whose kinetic energies depend on momenta but not coordinates[8]; the constraints in the double-pendulum system mean that the kinetic energy depends on $\theta_1 - \theta_2$. Consequently, symplectic integrators cannot be relied on here. While Runge-Kutta does not rigorously conserve energy, its fourth-order accuracy is sufficient to keep energy losses to a very low level (10^{-8} or smaller).

The adaptive step-size was implemented by advancing our simulation by 2 steps of size $\Delta t/2$, and also by 1 step of size Δt. The product of the 2 half-steps is taken as the more reliable result and used for the subsequent evolution of the system. However, an error is estimated by subtracting the energy computed from 2 half-steps from the energy computed

by advancing the system in a single full step. Although the error in 4th order Runge-Kutta is more often computed from changes in the variables themselves rather than functions of the variables, because the energy is a well-behaved function of the variables its error should still be of fifth order in Δt, just like the errors in the variables. This error ΔE is used to rescale the subsequent time step:

$$\Delta t' = \Delta t \left| \frac{\Delta E(\text{target})}{\Delta E} \right|^{0.2} \tag{1}$$

where $\Delta E(\text{target})$ is the desired accuracy. We typically worked with a target accuracy of one part in 10^8. We only ran the simulations in cases for which the starting energy was large enough to, in principle, allow a flip. However, we checked our code by independently running a small subset of the low-energy cases, and verifying that they did not flip during the run time of the simulation.

We implemented the algorithm on a Mac Tower with 16 processors, running in Ubuntu. Multiple simulations (with different m_2 and l_2 parameter values) were run simultaneously. We ran simulations for times up to 500, in units where small oscillations of the upper pendulum have period 1. We checked to make sure that the flip time was the same if we reversed both starting angles, and to make sure that energy was conserved (to within the desired accuracy) during the evolution of the system. After verifying that the flip time was unchanged by reversal of the angles for a small simulation (780 angles, results showin in Fig. 3), we did the rest of our simulations with $\theta_1 > 0$, to cut simulation time in half. While it is too coarse to be taken as verification of the figure from Wikipedia, it is at least consistent with that figure, and clearly shows the proper symmetry, validating our decision to proceed by studying just half of parameter space in the rest of this investigation. We considered 3 different cases: $l_2 = 1$ and $m_2 = 2$, $l_2 = 1$ and $m_2 = 1$, and $l_2 = 2$ and $m_2 = 1$. The only reason for not considering more cases is that obtaining even these results required 6 weeks of processor time.

As an additional check, we ran the simulation for small starting angles (much less than 1), and verified that we could observe the well-known[9] normal modes of the small-amplitude double pendulum. For small amplitudes and equal masses and lengths we found modes with periods of 0.54 and 1.3, and with θ_2 having an amplitude $\sqrt{2}$ larger than θ_1, exactly as predicted.

Once we had obtained plots of $\log t_{\text{flip}}$ *vs.* θ_1 and θ_2, we analyzed them in ImageJ[13],

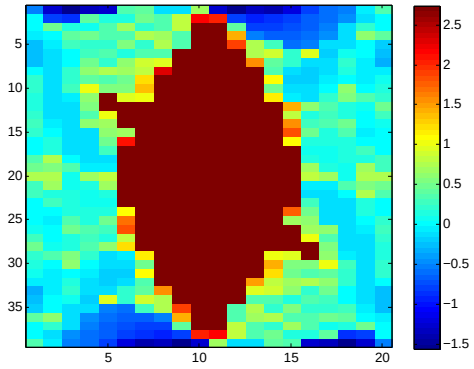

FIG. 3: Logarithm (base 10) of flip time versus starting angles for a double pendulum with equal masses and arm lengths. The numbering on the axes indicates pixel number, not angle. This coarse-grained simulation was performed only to check that the output had the proper symmetry. After verifying the symmetry in phase space, subsequent simulations were done with half as many starting angles, to save time.

an open-source image-processing package. We thresholded the images at different time cut-offs, to obtain simple black-and-white maps, and computed the fractal dimensions of those maps using the box-counting method[10], as fractal dimension is a useful parameter for characterizing the geometry of highly-branched patterns. Thresholding and box-counting were accomplished with ImageJ's built-in tools (**Image/Adjust/Threshold** and **Analyze/Tools/Fractal Box Count** in the ImageJ menus). We then converted the logarith-

mic images to 8 bit images (0-255), set thresholds of either 112, 128, or 144 (values outside that range had little detail), and determined the fractal dimension d_f of each thresholded image.

III. RESULTS

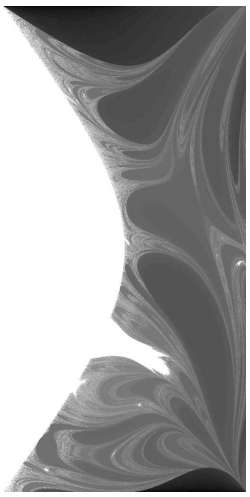

FIG. 4: Flip time *vs.* initial conditions $\theta_1(0)$ and $\theta_2(0)$ for parameters $l_2 = 1$ and $m_2 = 1$. The logarithm of time is displayed, ranging from -1.5607 (black) to $+2.7404$ (white).

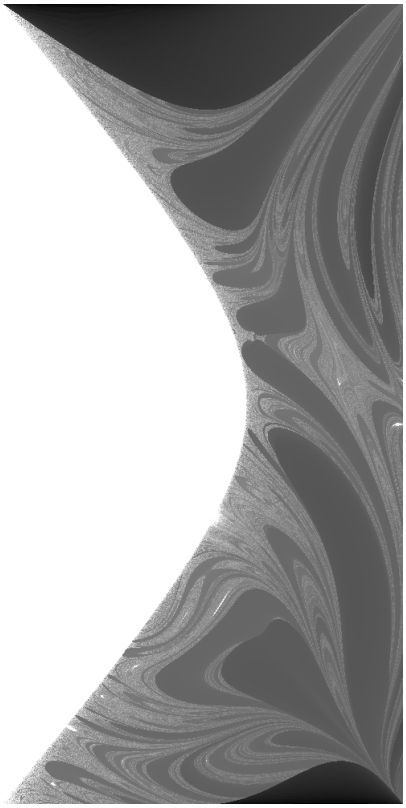

FIG. 5: Flip time *vs.* initial conditions $\theta_1(0)$ and $\theta_2(0)$ for parameters $l_2 = 1$ and $m_2 = 2$. The logarithm of time is displayed, ranging from -1.56 (black) to $+2.74$ (white).

TABLE I: Fractal dimension d_f of the $\log t_{\text{flip}}$ *vs.* starting angles plot for different thresholds and pendulum configuations (m_2 and l_2). Thresholds are relative to the logarithm of the maximum flip time.

Parameters		Threshold				
l_2	m_2	100	112	128	160	180
1	1	$d_f = 1.9050$	1.8543	1.7990	1.8220	1.8573
1	2	$d_f = 1.9145$	1.8751	1.8360	1.8400	1.8772
2	1	$d_f = 1.8533$	1.9212	1.9083	1.8915	1.8980

Plots of the logarithm of flip time *vs.* starting angles are shown in Figures 4 through 6. The range of θ_1 (horizontal) was 0 to $\pi - 0.01$, and the range of θ_2 was $-\pi + 0.01$ to $\pi - 0.01$. The effects of thresholding are illustrated in Fig. 7. All of the figures exhibit geometrical patterns strongly reminiscent of the Wikipedia figure, including an excluded region that is larger than the energy-forbidden region in the center of phase space. In Fig. 6 the forbidden

region is triangular rather than round, due to the fact that for $l_2 = 2$ and m_2 the equation for the forbidden region reduces to $\cos\theta_1 = -\cos\theta_2$, so $\theta_2 = \pm(\pi - \theta_1)$. In some of the figures there is also a pronounced protrusion of the forbidden region in its lower right corner, consistent with Fig. 2.

Fractal nature is apparent from a visual inspection of the images, but we verified this by computing fractal dimensions of thresholded images. Fractal dimensions are shown in Table I. All of the fractal dimensions are non-integer, confirming that these are indeed fractal patterns. Moreover, the dimensions are quite similar for all parameter values, supporting the proposition that this behavior is universal for double-pendulum systems in the chaotic regime. Finally, the fact that the fractal dimension is insensitive to thresholding supports the idea that there is a power-law distribution of flip times. If we apply a different threshold, which should have the same effect as changing the simulation time, we get a similar distribution of cases that flip, consistent with a power-law distribution of flip times: Rescaling the variables produces quantitative but not qualitative changes in power-law distributions.

Additionally, the fractal dimensions fall in the range 1.8 to 1.9. For comparison, we computed the fractal dimension of a gray-scale version of Fig. 2, and found d_f values between 1.8 and 1.9, depending on the threshold. Thus, although our figures do not bear an exact resemblance to Fig. 2, their fractal characteristics are similar to those of the Wikipedia figure.

IV. SUMMARY

This work leads to three key findings. First, in a double-pendulum system, the time required for the bottom pendulum to "flip" over depends very sensitively on the initial conditions, as demonstrated by the fractal nature of our plots of t_{flip} as a function of initial conditions. In particular, the flip times obey a power law distribution, demonstrated by the fact that the fractal dimension changes very little when we change the threshold applied to the plots. This is a common phenomenon in chaotic systems. Second, the distribution of flip times is similar even when we change the specifics of the pendulum, *e.g.* varying the mass and length ratios, or changing the system from two point masses to two rods with their mass distributed along the length. This is confirmed by the fact that the fractal dimensions of our flip-time plots are all between 1.8 and 1.9.

Our third finding, of broader significance than chaos theory, is that we should not automatically discount a result posted on Wikipedia without citation. This is not to say that an uncited report should be trusted without further scrutiny. Rather, its plausibility should be evaluated like any other claim and then, if plausible, be subjected to further investigation, as we have done here. Admittedly, we could have made that general statement even without performing these calculations, but the fact that we were able to verify a result on the community-edited Wikipedia, while finding an error (see Appendix) at Wolfram Scienceworld (which follows a more traditional model of publishing, based on the reputation of a single named writer), should temper the common academic distrust of Wikipedia. Wikipedia articles should not be trusted, per se (all scientific claims require sourcing and testing), but they deserve an open-minded reception.

APPENDIX A: APPENDIX: DERIVATION OF EQUATIONS

We will derive the equations of motion in the Hamiltonian form. Although we did not ultimately use the Euler-Cromer algorithm (and hence did not, strictly speaking, need our equations in Hamiltonian form), the Hamiltonian formalism gives us 4 first-order equations, whereas the Lagrangian formalism would give two coupled second-order equations. Even if we changed to variables ω and θ, we would have two equations that both involve $\dot{\omega}_1$ and $\dot{\omega}_2$. Disentangling these equations would be almost as much work as writing down a Hamiltonian, but would be less physically transparent.

We assume that the masses are concentrated at two points, for simplicity, rather than treating 2 rigid rods. While this differs slightly from the system described on the Wikipedia page that we are auditing, the qualitative behavior should be similar, and the equations are slightly easier to derive for the case of 2 point masses. The coordinates of the masses in our double pendulum system are:

$$x_1 = l_1 \sin\theta_1 \tag{A.1}$$

$$x_2 = x_1 + l_2 \sin\theta_2 \tag{A.2}$$

$$y_1 = -l_1 \cos\theta_1 \tag{A.3}$$

$$y_2 = y_1 - l_2 \cos\theta_2 \tag{A.4}$$

where l_1 and l_2 are the lengths of the rods. The velocities are then:

$$v_{x1} = l_1\dot{\theta_1}\cos\theta_1 \tag{A.5}$$

$$v_{x2} = v_{x1} + l_2\dot{\theta_2}\cos\theta_2 \tag{A.6}$$

$$v_{y1} = l_1\dot{\theta_1}\sin\theta_1 \tag{A.7}$$

$$v_{y2} = v_{y1} + l_2\dot{\theta_2}\sin\theta_2 \tag{A.8}$$

We will work in units where $m_1 = l_1 = 1$, and m_2 and l_2 are thus dimensionless. The kinetic energy is thus:

$$KE = \frac{1}{2}\left((1+m_2)\dot{\theta_1}^2 + m_2 l_2^2\dot{\theta_2}^2 + 2m_2 l_2\dot{\theta_1}\dot{\theta_2}\cos(\theta_1-\theta_2)\right) \tag{A.9}$$

For potential energy, we pick our units of time so that $g = 4\pi^2$, making the frequency of small oscillations of the top pendulum 2π (*i.e.* period of 1). The potential energy is then:

$$U = -4\pi^2(\cos\theta_1 + m_2 l_2(\cos\theta_1 + \cos\theta_2)) \tag{A.10}$$

If we take the difference of kinetic and potential energy, the Lagrangian is:

$$L = \frac{1}{2}\left((1+m_2)\dot{\theta_1}^2 + m_2 l_2^2\dot{\theta_2}^2 + 2m_2 l_2\dot{\theta_1}\dot{\theta_2}\cos(\theta_1-\theta_2)\right) + 4\pi^2(\cos\theta_1 + m_2 l_2(\cos\theta_1 + \cos\theta_2)) \tag{A.11}$$

The canonical momenta are:

$$p_1 = (1+m_2)\dot{\theta_1} + m_2 l_2\dot{\theta_2}\cos(\theta_1-\theta_2) \tag{A.12}$$

$$p_2 = m_2 l_2^2\dot{\theta_2} + m_2 l_2\dot{\theta_1}\cos(\theta_1-\theta_2) \tag{A.13}$$

In order to construct a Hamiltonian we need to solve for $\dot{\theta_1}$ and $\dot{\theta_2}$ in terms of p_1 and p_2. We have a system of two linear equations and two unknowns. Solving this system of equations gives:

$$\dot{\theta_1} = \frac{l_2 p_1 - p_2\cos(\theta_1-\theta_2)}{l_2(1+m_2-m_2\cos^2(\theta_1-\theta_2))} \tag{A.14}$$

$$\dot{\theta_2} = \frac{(1+m_2)p_2 - l_2 m_2 p_1\cos(\theta_1-\theta_2)}{m_2 l_2^2(1+m_2-m_2\cos^2(\theta_1-\theta_2))} \tag{A.15}$$

Next we express L in terms of the momenta, by rewriting the time derivatives in terms of momenta:

$$L = \frac{l_2^2 m_2 p_1^2 + (1+m_2)p_2^2 - 2l_2 m_2 p_1 p_2\cos(\theta_1-\theta_2)}{l_2^2 m_2\left(2+m_2-m_2\cos(\theta_1-\theta_2)\right)} + 4\pi^2(\cos\theta_1 + m_2 l_2(\cos\theta_1 + \cos\theta_2)) \tag{A.16}$$

Now we can construct a Hamiltonian $H = p_1\dot{\theta_1} + p_2\dot{\theta_2} - L$.

$$H = \frac{l_2^2 m_2 p_1^2 + (1+m_2)p_2^2 - 2l_2 m_2 p_1 p_2 \cos(\theta_1 - \theta_2)}{2l_2^2 m_2 \left(1 + m_2 \sin^2(\theta_1 - \theta_2)\right)} \\ -4\pi^2 \left(\cos\theta_1 + m_2 l_2 (\cos\theta_1 + \cos\theta_2)\right) \tag{A.17}$$

Note that the kinetic energy depends on the difference $\theta_1 - \theta_2$, but not on the linearly independent coordinate $\theta_1 + \theta_2$. This comes from the fact that the kinetic energy depends on (1) how quickly the orientation of the system is changing (encoded in the derivatives of the angles) and (2) how the mass is distributed in the system (encoded in the difference of the angles), but not on the orientation itself. Consequently, $\frac{\partial H}{\partial \theta_1} + \frac{\partial H}{\partial \theta_2} = 0$, somewhat simplifying the code.

The messy part is computing derivatives of H to get time derivatives of coordinates and momenta:

$$\dot{\theta_1} = \frac{\partial H}{\partial p_1} = \frac{l_2 p_1 - p_2 \cos(\theta_1 - \theta_2)}{l_2(1 + m_2 \sin^2(\theta_1 - \theta_2))} \tag{A.18}$$

$$\dot{\theta_2} = \frac{\partial H}{\partial p_2} = \frac{(1+m_2)p_2 - l_2 m_2 p_1 \cos(\theta_1 - \theta_2)}{l_2^2 m_2 (1 + m_2 \sin^2(\theta_1 - \theta_2))} \tag{A.19}$$

$$\begin{aligned} \dot{p_1} &= -\frac{\partial H}{\partial \theta_1} \\ &= -\frac{(-(1+m_2)p_2 + l_2 m_2 p_1 \cos(\theta_1 - \theta_2))\,(-l_2 p_1 + p_2 \cos(\theta_1 - \theta_2))}{l_2^2 \left(1 + m_2 \sin^2(\theta_1 - \theta_2)\right)^2} \sin(\theta_1 - \theta_2) \\ &- 4\pi^2 (1 + m_2 l_2) \sin\theta_1 \end{aligned} \tag{A.20}$$

$$\begin{aligned} \dot{p_2} &= -\frac{\partial H}{\partial \theta_2} \\ &= \frac{(-(1+m_2)p_2 + l_2 m_2 p_1 \cos(\theta_1 - \theta_2))\,(-l_2 p_1 + p_2 \cos(\theta_1 - \theta_2))}{l_2^2 \left(1 + m_2 \sin^2(\theta_1 - \theta_2)\right)^2} \sin(\theta_1 - \theta_2) \\ &- 4\pi^2 m_2 l_2 \sin\theta_2 \end{aligned} \tag{A.21}$$

Interestingly, while checking this derivation we discovered a small error in similar work at Wolfram Scienceworld[14]. Specifically, equation 29 on the Wolfram site is missing a 2 in the last term of the numerator. It is fascinating that a page created by a known, credentialed

source had a mistake that was found in this investigation, while our results confirmed a claim found on a site that anybody can edit.

---

* arsmall@csupomona.edu

[1] J. Gleick, *Chaos: Making a new science* (Open Road Media, 2011).

[2] M. R. Schroeder, *Fractals, chaos, power laws: Minutes from an infinite paradise* (Courier Dover Publications, 2012).

[3] R. Levien and S. Tan, American Journal of Physics **61**, 1038 (1993).

[4] R. Cross, American Journal of Physics **73**, 330 (2005).

[5] N. J. Harmon, C. Leidel, and J. F. Lindner, American Journal of Physics **71**, 871 (2003).

[6] W. H. Press, *Numerical recipes: the art of scientific computing* (Cambridge University Press, 2007), 3rd ed.

[7] T. Timberlake and J. E. Hasbun, American Journal of Physics **76**, 334 (2008).

[8] R. I. McLachlan and P. Atela, Nonlinearity **5**, 541 (1992).

[9] J. R. Taylor, *Classical mechanics* (University Science Books, 2005).

[10] T. Smith, G. Lange, and W. Marks, Journal of neuroscience methods **69**, 123 (1996).

[11] http://scienceworld.wolfram.com/physics/DoublePendulum.html

[12] Since Wikipedia articles can be edited, instead of citing the original page we are citing a snapshot that was taken on October 12, 2013 and is archived at http://www.webcitation.org/6KKM4Ly3i

[13] http://rsbweb.nih.gov/ij/

[14] The page is at http://scienceworld.wolfram.com/physics/DoublePendulum.html. As of October 20, 2013 the error is still there. However, it is expected that this error will be fixed. Unfortunately, Wolfram's site does not allow archiving by webcitation.org.

FIG. 6: Flip time *vs.* initial conditions $\theta_1(0)$ and $\theta_2(0)$ for parameters $l_2 = 2$ and $m_2 = 1$. The logarithm of time is displayed, ranging from $-\infty$ (black) to $+$ (white).

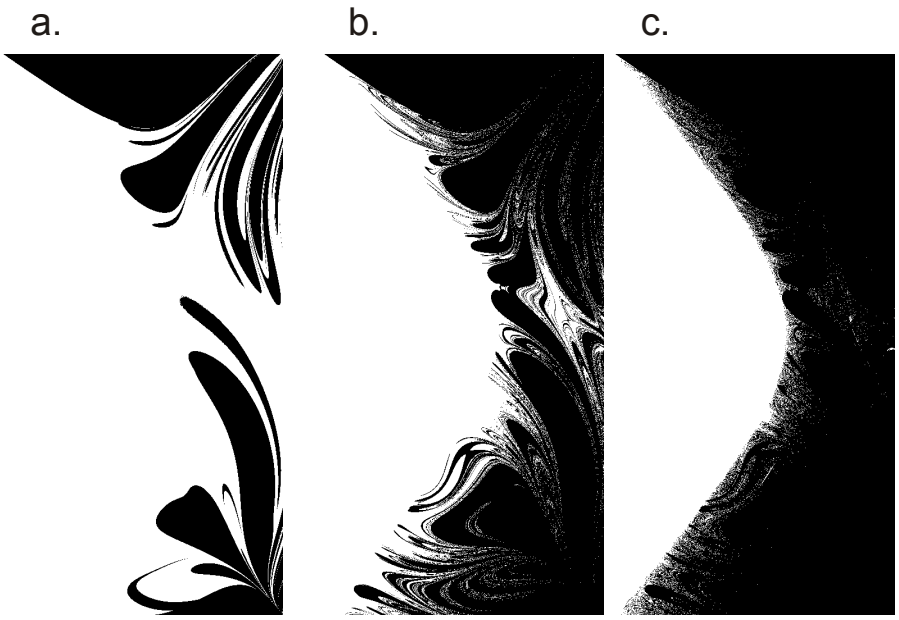

FIG. 7: Thresholded versions of Fig. 5, with thresholds of either (a) 100, (b) 128, or (c) 160 (logarithm of flip time). Fractal dimensions, determined via the box-counting method in ImageJ, are 1.9145, 1.8360, and 1.8400 for (a), (b), and (c) respectively.